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Multiplicities of conjugacy class sizes of finite groups

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ABSTRACT

It has been proved recently by Moretó (2007) [8] and Craven (2008) [3] that the order of a finite group is bounded in terms of the largest multiplicity of its irreducible character degrees. A conjugacy class version of this result was proved for solvable groups by Jaikin-Zapirain (2005) [7]. In this note, we prove that if G is a finite simple group then the order of G, denoted by $|G|$, is bounded in terms of the largest multiplicity of its conjugacy class sizes. As a consequence, we prove that if the largest multiplicity of conjugacy class sizes of any quotient of a finite group G is m, then $|G|$ is bounded in terms of m.

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1. Introduction

Let G be a finite group. The multiplicity of a (complex) character degree of G is the number of distinct irreducible characters of that degree of the group G. Moretó conjectures in [8] that if the largest multiplicity of a character degree of G is m, then the order of G is m-bounded. Here, we say that $|G|$ is *m-bounded* or *bounded in terms of m* if $|G| < f(m)$ for some real-valued function f on $\mathbb{N}$. In the same paper, Moretó, by using the classification of finite simple groups, proves that the conjecture is true for all finite groups if it is true for the symmetric groups. The problem for the symmetric groups is naturally combinatoric and was done by Craven in [3].

Theorem A. *(See Craven [3], Moretó [8].) If the largest multiplicity of irreducible ordinary character degrees of a finite group G is m, then $|G|$ is m-bounded.*

In a similar way, we define the multiplicity of a conjugacy class size of G to be the number of distinct conjugacy classes of that size. Because of the natural duality between the characters and conjugacy classes, one might expect a conjugacy class version of Theorem A:

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Conjecture B. *If the largest multiplicity of conjugacy class sizes of a finite group G is m, then $|G|$ is m-bounded.*

We would like to point out that this problem is not new, at least for solvable groups. Indeed, it has been proved by Jaikin-Zapirain for nilpotent groups in [6] and then for solvable groups in [7].

Our first result of this note is the proof of Conjecture B for finite simple groups.

Theorem C. *Let G be a finite simple group. If the largest multiplicity of conjugacy class sizes of G is m, then $|G|$ is m-bounded.*

Using Theorem C and drawing upon some techniques of Moretó in [8], we prove a weaker version of Conjecture B.

Theorem D. *If the largest multiplicity of conjugacy class sizes of any quotient of a finite group G is m, then $|G|$ is m-bounded.*

Notation. Let X be a finite group and $x \in X$. We denote the centralizer of x in X by $C_X(x)$, the conjugacy class of x in X by $\mathrm{cl}_X(x)$, and the size of this class by $|\mathrm{cl}_X(x)|$. Other notations are standard.

2. Proof of Theorem C

The aim of this section is to prove Theorem C. The case of cyclic groups of prime orders is obvious. Therefore, it remains to consider finite simple groups of Lie type and alternating groups.

Lemma 2.1. *Theorem* C *holds for finite simple groups of Lie type.*

Proof. Let $G_n(q)$ be a simple group of Lie type (twisted or untwisted) of rank n defined over a field of q elements and let $k^*(G_n(q))$ denote the number of orbits of $\mathrm{Aut}(G_n(q))$ acting on $G_n(q)$. Lemma 6.3 of [8] claims that

$$\frac{k^*(G_n(q))}{d(|\mathrm{Aut}(G_n(q))|)} \to \infty \quad \text{as } q \to \infty,$$

where $d(|\mathrm{Aut}(G_n(q))|)$ is the number of positive divisors of $|\mathrm{Aut}(G_n(q))|$. Since every conjugacy class size of $G_n(q)$ divides $|G_n(q)|$ and the number of conjugacy classes of $G_n(q)$ is at least $k^*(G_n(q))$, it follows that the average multiplicity of conjugacy class sizes of $G_n(q)$ tends to ∞ as q tends to ∞ and therefore we are done for the case of exceptional simple groups, where the rank n is $\leqslant 8$.

Assume that $G_n(q)$ is a simple classical group. By a result of Malle (see Corollary 3.4 of [8]), $G_n(q)$ has a semisimple element s of order at least $(q^{[n+1/2]} - 1)/(n+1)$ conjugate to at most $2n+1$ of its powers (actually, for the orthogonal groups, Malle states only the results for projective special orthogonal groups but the same arguments work for the simple groups). Note that s and s^t have the same centralizers whenever t is coprime to $|s|$. It follows that $G_n(q)$ has at least

$$\frac{\min_{k \geqslant (q^{[n+1/2]}-1)/(n+1)} \varphi(k)}{2n+1}$$

conjugacy classes (of semisimple elements) of the same size, where φ is the Euler's totient function. It is well known that $\varphi(k) > \sqrt{k/2}$ for any $k > 0$. Therefore, the above quotient tends to ∞ as $|G_n(q)|$ tends to ∞, as desired. $\square$

In the following lemma, S_n and A_n are the symmetric and alternating groups, respectively, of degree n.

Lemma 2.2. *For every positive integer M, there exists a positive integer $N = N(M)$ such that for every $n \geqslant N$, A_n has M conjugacy classes of the same size which are invariant under S_n.*

Proof. It is known that conjugacy classes of the symmetric group S_n of degree n are in one-to-one correspondence with partitions of size n. Let us recall some standard notation and terminology of partitions. A partition λ of size n is a finite sequence $(\lambda_1, \lambda_2, \ldots)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ and $\lambda_1 + \lambda_2 + \cdots = n$. Each λ_i is called a part of λ. Let $m_i(\lambda)$ denote the number of parts of λ of size i. We say that λ is even (odd, resp.) if permutations of cycle type λ are even (odd, resp.). We will often denote the centralizer size of a permutation (in S_n) of cycle type λ by the centralizer size of λ and denote it by $C(\lambda)$. It is well known that

$$C(\lambda) = \prod_i \bigl(i^{m_i(\lambda)}\bigr)\bigl(m_i(\lambda)\bigr)!.$$

In particular, if all the parts of λ are different then $C(\lambda)$ is product of those parts.

We will prove that for any M, there exists an integer $N = N(M)$ big enough so that for each $n \geqslant N$ there are M even partitions of size n with the same centralizer size and the conjugacy classes of S_n associated with these partitions are indeed conjugacy classes of A_n.

If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition and $\lambda_0 \geqslant \lambda_1$, we define the partition

$$(\lambda_0, \lambda) := (\lambda_0, \lambda_1, \lambda_2, \ldots).$$

It is clear that

$$C\bigl((\lambda_0, \lambda)\bigr) = \lambda_0 C(\lambda) \quad \text{if } \lambda_0 > \lambda_1. \tag{2.1}$$

For each $i \in \mathbb{N}$, let (a_i, b_i, c_i) be either $(10, 9, 1)$ or $(15, 3, 2)$. Then (a_i, b_i, c_i) is a partition of size 20 with centralizer size 90. Choose an odd integer k so that $2^k > M$. Consider the set P of 2^k odd partitions of the form

$$\bigl(21^{k-1}a_k, 21^{k-1}b_k, 21^{k-1}c_k, \ldots, 21a_2, 21b_2, 21c_2, a_1, b_1, c_1\bigr).$$

All partitions in P have same size $20(21^{k-1} + 21^{k-2} + \cdots + 1) = 21^k - 1$ and centralizer size $90^k \cdot 21^{3k(k-1)/2}$. Therefore, if n is an even integer bigger than $21^k + 21^{k-1}15$, we obtain 2^k even partitions of the form $(n - |\lambda|, \lambda)$ with $\lambda \in P$, all of size n and centralizer size $(n - |\lambda|)C(\lambda)$ by (2.1).

To handle odd n, we consider the set P' of 2^{k+1} even partitions of the form

$$\bigl(21^k a_{k+1}, 21^k b_{k+1}, 21^k c_{k+1}, \ldots, 21a_2, 21b_2, 21c_2, a_1, b_1, c_1\bigr).$$

As before, all partitions in P' have the same size $21^{k+1} - 1$ and centralizer size $90^{k+1} \cdot 21^{3k(k+1)/2}$. Hence, if n is an odd integer bigger than $21^{k+1} + 21^k 15$, we obtain 2^{k+1} even partitions of the form $(n - |\lambda|, \lambda)$ with $\lambda \in P'$.

Since all partitions we have created have an even part, the conjugacy classes of S_n associated to them are indeed conjugacy classes of A_n. The lemma is completely proved. $\square$

The following is an immediate consequence of Lemma 2.2.

Corollary 2.3. *Theorem* C *holds for the alternating groups.* $\square$

3. Proof of Theorem D

We collect some results needed in the proof of Theorem D. The following number-theoretic result is in [4] and is stated as Lemma 4.1 in [8].

Lemma 3.1. (i) $\lim_{n\to\infty} \frac{d(n!)}{2^{\frac{c\log n!}{(\log\log n!)^2}}} = 1$, *where c is a constant and* $d(n!)$ *is the number of divisors of* $n!$.

(ii) $\lim_{k\to\infty} \frac{a^{ck}}{(k+1)d(k!)} = \infty$, *where c is a positive constant and a is a constant bigger than* 1. □

The next result is due to Babai and Pyber [1] and appears as Theorem 4.3 in [8].

Lemma 3.2. *Let G be a permutation group on a set Ω of cardinality k and assume that G does not contain any alternating group larger than A_n as a composition factor. Then the number of orbits of G on the power set $\mathcal{P}(\Omega)$ is at least $a^{k/n}$ where a is some constant bigger than* 1. □

As mentioned earlier, Conjecture B has been proved for solvable groups by Jaikin-Zapirain [7]. We need this result in the proof of Theorem D.

Theorem 3.3. *(See Jaikin-Zapirain [7].) If the largest multiplicity of conjugacy class sizes of a finite solvable group G is m, then the order of G is m-bounded.* □

The following lemma is a nice property of the action of the automorphism group of a simple group on its conjugacy classes which might be of independent interest.

Lemma 3.4. *Let S be a finite non-abelian simple group. Then there is a nontrivial conjugacy class of S invariant under the conjugate action of* $\mathrm{Aut}(S)$.

Proof. Checking the Atlas [2] case by case, we see that every conjugacy class of involutions in a simple sporadic group or the Tits group is invariant under the automorphism group. It is easy to see that the same conclusion also holds for alternating groups. So we can assume that S is a finite simple group of Lie type.

1) First we prove the lemma for Suzuki and Ree groups. In this case, $\mathrm{Out}(S)$ is a cyclic group of field automorphisms of S. Therefore, the action of $\mathrm{Out}(S)$ on conjugacy classes is permutation isomorphic to its action on irreducible characters of S. Since the *Steinberg character* of S is invariant under $\mathrm{Out}(S)$, there must be a nontrivial conjugacy class of S invariant under $\mathrm{Out}(S)$.

2) Next we assume that S is $F_4(2^n) = F_4(q)$. The conjugacy classes of S is described in [9]. Inspecting the list of class representatives and the orders of their centralizers in [9, Theorem 2.1], we see that S has a unique conjugacy classes of involutions with order of its representative centralizer $q^{24}(q^2-1)(q^4-1)$. This class is therefore invariant under $\mathrm{Aut}(S)$.

3) Now we handle the case where S is a finite simple group of Lie type in even characteristic but not the ones considered in 1) and 2). Since $B_n(2^m) \simeq C_n(2^m)$, we can assume that S is not of type B_n in even characteristic. As claimed in [5, p. 103], the class of *long root subgroups* of S is invariant under $\mathrm{Aut}(S)$ and all *long root elements*, which are nonidentity elements of long root subgroups, are S-conjugate. This means that the unique conjugacy class of long root elements of S is invariant under $\mathrm{Aut}(S)$.

4) Finally, we assume that S is a finite group of Lie type in odd characteristic. Table 4.5.1 of [5] describes all conjugacy classes of involutions of S including the structures of centralizers of class representatives. In general, these centralizers is "close" (cf. the interpretation of [5, Table 4.5.1]) to a central product of finite groups of Lie type. For each type of S, we choose a conjugacy class so that the structure of corresponding centralizer is different from that of other centralizers. This class is therefore invariant under $\mathrm{Aut}(S)$. The conjugacy class with representative t_1 or t_1' will work in all cases except $S = {}^3D_4(q)$ where we choose the class with representative t_2. □

Proof of Theorem D. The proof is divided into the following steps:

1) If A_n is a composition factor of G, then n is m-bounded.

Proof. Suppose that N/M is a chief factor of G that is a direct product of copies of A_n. Then N/M is also a chief factor of G/M. Since G/M satisfies the hypothesis of the theorem, we can assume that $M = 1$. The subgroup N is then a minimal normal subgroup of G and we assume that N is the direct product of k copies of A_n. The quotient $H = G/C_G(N)$ is embedded in $\mathrm{Aut}(N) = \mathrm{Aut}(A_n) \wr S_k = S_n \wr S_k$. Since $N \cong (N \times C_G(N))/C_G(N)$, we can consider N as a subgroup of H.

Assume the contrary that n is not m-bounded. Lemma 2.2 then implies that A_n can have arbitrarily many conjugacy classes of the same size and invariant under S_n. Let's pick n big enough so that A_n has $m+1$ such conjugacy classes $K_1, K_2, \dots, K_{m+1}$ all of size s. We then obtain $m+1$ corresponding conjugacy classes of N of size s^k: $K_i' = K_i \times K_i \times \cdots \times K_i$, $i = 1, 2, \dots, m+1$. Since each K_i is invariant under S_n, K_i' is also invariant under $S_n \wr S_k$. In particular, the K_i''s are invariant under H as $H \leqslant S_n \wr S_k$. Therefore, H has $m+1$ conjugacy classes of size s^k, which is a contradiction.

2) Let $O_\infty(G)$ denote the maximal normal solvable subgroup of G. We claim that if $|G : O_\infty(G)|$ is m-bounded then $|G|$ is m-bounded. Let x be an arbitrary element in $O_\infty(G)$. Then

$$\frac{|\mathrm{cl}_G(x)|}{|\mathrm{cl}_{O_\infty(G)}(x)|} = \frac{|G|}{|O_\infty(G)|} \cdot \frac{|C_{O_\infty(G)}(x)|}{|C_G(x)|},$$

which divides $|G : O_\infty(G)|$. Since the multiplicity of every conjugacy class size of G is less than or equal to m and $|G : O_\infty(G)|$ is m-bounded, it follows that the multiplicity of any conjugacy class size of $O_\infty(G)$ is m-bounded. Using Theorem 3.3, we see that $|O_\infty(G)|$ is m-bounded and therefore $|G| = |O_\infty(G)||G : O_\infty(G)|$ is m-bounded.

3) Now we show that $|G : O_\infty(G)|$ is m-bounded. Since any quotient of $G/O_\infty(G)$ is a quotient of G, the group $G/O_\infty(G)$ satisfies the hypothesis of the theorem. Therefore, we may assume that $O_\infty(G) = 1$. Let $\mathrm{Soc}(G)$ denote the socle of G. In other words, $\mathrm{Soc}(G)$ is the direct product of minimal normal subgroups of G, each of which is a direct product of non-abelian simple groups. The group G now is embedded into $\mathrm{Aut}(\mathrm{Soc}(G))$. Therefore, it suffices to show that $|\mathrm{Soc}(G)|$ is m-bounded. This is done by the following two steps.

4) If S is a direct factor of $\mathrm{Soc}(G)$, then the number of times that S appears in $\mathrm{Soc}(G)$ is m-bounded.

Proof. Suppose that the number of times that S appears in $\mathrm{Soc}(G)$ is k. Let N be the direct product of k copies of S. Then N is a normal subgroup of G. As in step 1, $H = G/C_G(N)$ is embedded in $\mathrm{Aut}(N) = \mathrm{Aut}(S) \wr S_k$. Set $K := H \cap \mathrm{Aut}(S)^k$. Note that N can be viewed as a subgroup of H and therefore N is a subgroup of K. Also, H/K is a permutation group on k letters. By step 1, there is an integer n which is bounded in terms of m such that H/K does not contain any alternating group larger than A_n. Therefore, by Lemma 3.2, the number of orbits of H/K on the power set of k letters is at least $a^{k/n}$ where $a > 1$ is some constant. It follows that there exists $l \in \{0, 1, 2, \dots, k\}$ such that the number of orbits of H/K on the subsets of cardinality l is at least $a^{k/n}/(k+1)$.

Following Lemma 3.4, let C be a nontrivial conjugacy class of size s of S invariant under $\mathrm{Aut}(S)$. Then C is also a conjugacy class of $\mathrm{Aut}(S)$. Consider $\binom{k}{l}$ conjugacy classes of K which are products of l copies of C and $k-l$ copies of the trivial conjugacy class. From the conclusion of the previous paragraph, we obtain that there are at least $a^{k/n}/(k+1)$ H-orbits of conjugacy classes of K of size s^l. Therefore, H has at least $a^{k/n}/(k+1)d(k!)$ conjugacy classes of the same size. This means that $a^{k/n}/(k+1)d(k!) \leqslant m$. Lemma 3.1 then implies that k is bounded in terms of m, as wanted.

5) If S is a direct factor of $\mathrm{Soc}(G)$, then $|S|$ is m-bounded.

Proof. Define k, N, H, and K as in step 4. We then have that H/K is a permutation group on k letters and therefore H/K is m-bounded. It follows that the largest multiplicity of conjugacy class sizes of K/N is bounded in terms of m since $H/N = G/NC_G(N)$ does not have more than m conjugacy classes of the same size. Note that K/N is isomorphic to a subgroup of $\mathrm{Out}(S)^k$, which is solvable by Schreier's conjecture. So K/N is solvable and hence its order is m-bounded by Theorem 3.3. We then have that $|H:N|$ is m-bounded.

Recall that the largest multiplicity of conjugacy class sizes of H is not more than m. It follows that the largest multiplicity of conjugacy class sizes of N is bounded in terms of m and therefore the same thing happens to S. Using Theorem C, we obtain that $|S|$ is m-bounded. □

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